On hypermatrices whose blocks are commutable in pairs and their application in lattice-dynamics.

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Notations.

$\mathbf{A} = [a_{ij}]$ = matrix composed of the scalars a_{ij}

$[\mathbf{A}_{ij}]$ = hypermatrix composed of the blocks $\mathbf{A}_{ij}$

$\mathbf{A}^*$ = conjugate transpose of $\mathbf{A}$

$\mathbf{a}, \mathbf{a}_i, \ldots$ = column vectors

$\mathbf{b}^*, \mathbf{b}^j, \ldots$ = row vectors

$[\mathbf{a}_i \mathbf{b}^j]$ = hypermatrix composed of the blocks $\mathbf{a}_i \mathbf{b}^j$

$\mathbf{A} \cdot \times \mathbf{B} = [\mathbf{A} b_{ij}]$; $[b_{ij}] = \mathbf{B}$

$\langle a_k \rangle = \langle a_1, a_2, \ldots, a_n \rangle$ = diagonal matrix

$\mathbf{E}_n$ = n-th order unit matrix

$|\mathbf{A}| = \det \mathbf{A}$ = determinant of $\mathbf{A}$

1. Introduction. It is known that the operational rules for hypermatrices composed of quadratic blocks and those for ordinary matrices composed of scalar elements nearly agree, the only difference between them being that the blocks are in general non-commutable. Therefore it is to be expected that concepts and computational methods referring to ordinary matrices can be extended to hypermatrices whose blocks are commutable in pairs.

The assumption of the commutability obviously implies that the blocks constitute a commutative ring, consequently rational scalar identities remain valid if the scalar indeterminates will be replaced by the blocks.

This observation suggests that the concepts determinant, minor, adjoint can be extended to hypermatrices whose blocks are commutable in pairs and that in this way certain rational operations with nm-th order matrices can be reduced to manipulation with n-th and m-th order matrices only.

Concerning the spectral decomposition of a hypermatrix composed of commutable blocks the following theorems are known.

The characteristic roots of the direct product $\mathbf{A} \cdot \times \mathbf{B}$ are the $n \cdot m$ products $a_i b_j$ of a characteristic root of $\mathbf{A}$ by one of $\mathbf{B}$. This is a special case of the following theorem of WILLIAMSON[1]): If the n^2 blocks $f_{ij}(\mathbf{A})$ of a

[1]) J. WILLIAMSON, The latent roots of a matrix of special type, *Bulletin American Math. Soc.*, **37** (1931), 585—590.

hypermatrix are arbitrary polynomials of the same matrix $\mathbf{A}$ and if $\mathbf{A}$ has the characteristic roots $a_1, a_2, \ldots, a_m$, then the characteristic roots of the hypermatrix $[f_{ij}(\mathbf{A})]$ are the $n \cdot m$ characteristic roots of the matrices

$$[f_{ij}(a_1)], [f_{ij}(a_2)], \ldots, [f_{ij}(a_m)].$$

These theorems suggest that not only the characteristic roots but the characteristic vectors too, hence the whole spectral decomposition of the hypermatrix $[f_{ij}(\mathbf{A})]$ can be built up from those of certain n-th resp. m-th order matrices by explicit and elementary operations.

2. Theorems. Both suggestions turn out to be justified and the main object of this paper will be the proof, discussion and application of the following two theorems.

Theorem. A. *If the m-th order blocks* $\mathbf{A}_{ij}$ *of the hypermatrix* $[\mathbf{A}_{ij}]$ *are commutable in pairs, then the determinant, resp. the adjoint of* $[\mathbf{A}_{ij}]$, *are given by the following formulae*

$$\det[\mathbf{A}_{ij}] = \det(\mathbf{det}[\mathbf{A}_{ij}]), \tag{1}$$

$$\operatorname{adj}[\mathbf{A}_{ij}] = \mathbf{adj}[\mathbf{A}_{ij}] \cdot \{\operatorname{adj}(\mathbf{det}[\mathbf{A}_{ij}]) \cdot \times \mathbf{E}_n\}, \tag{2}$$

where

$$\mathbf{det}[\mathbf{A}_{ij}] = \sum_{(\nu)} \pm \mathbf{A}_{1\nu_1} \mathbf{A}_{2\nu_2} \ldots \mathbf{A}_{n\nu_n}$$

and $\mathbf{adj}[\mathbf{A}_{ij}]$ *denotes a hypermatrix whose blocks depend in the same way upon the blocks* $\mathbf{A}_{ij}$, *as the elements of an ordinary adjoint* $\operatorname{adj}[a_{ij}]$ *depend upon the scalar elements* a_{ij}.

Theorem B. *If* $\mathbf{A}$ *is a (hermitical) symmetrical m-th order matrix with the characteristic roots* $a_1, a_2, \ldots, a_m$ *and if* $f_{ij}(x)$ $(i, j = 1, 2, \ldots, n)$ *are arbitrary polynomials of the real variable x, subject only to the restriction* $f_{ji}(x) = \overline{f_{ij}(x)}$, *then the spectral decomposition of the hypermatrix* $[\mathbf{A}_{ij}]$ *with the blocks* $\mathbf{A}_{ij} = f_{ij}(\mathbf{A})$ *is given by*

$$[\mathbf{A}_{ij}] = \mathbf{T}^* \langle \lambda_{11}, \ldots, \lambda_{1n}, \ldots, \lambda_{m1}, \ldots, \lambda_{mn} \rangle \mathbf{T}, \quad \mathbf{T}^*\mathbf{T} = \mathbf{E}_{mn}, \tag{3}$$

where the characteristic roots λ_{ij} *and the factors of the transforming matrix*

$$\mathbf{T} = \langle \overbrace{\mathbf{U}, \mathbf{U}, \ldots, \mathbf{U}}^{n} \rangle \mathbf{P} \langle \mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n \rangle \tag{3.1}$$

are to be found from the spectral decompositions

$$\mathbf{A} = \mathbf{U} \langle a_1, \ldots, a_m \rangle \mathbf{U}^*, \quad [f_{ij}(a_k)] = \mathbf{V}_k \langle \lambda_{k1}, \ldots, \lambda_{kn} \rangle \mathbf{V}_k^* \quad (k = 1, 2, \ldots, m) \tag{3.2}$$

and $\mathbf{P}$ *is the permutation matrix which transforms the sequence of ordered pairs*

$$(11)(12)\ldots(1m)(21)(22)\ldots(2m)\ldots(n1)(n2)\ldots(nm)$$

into the sequence

$$(11)(21)\ldots(n1)(12)(22)\ldots(n2)\ldots(1m)(2m)\ldots(nm).$$

The caracteristic vectors of $[\mathbf{A}_{ij}]$ are then the columns of $\mathbf{T}$.

The special case of Theorem B concerning the direct product $\mathbf{A}\cdot\times\mathbf{B}$ of the symmetrical matrices $\mathbf{A}$ and $\mathbf{B}$ seems worth to be formulated explicitely as

Corollary I. *The characteristic roots of* $\mathbf{A}\cdot\times\mathbf{B}$ *are the products* $a_i b_j$ *of the characteristic roots of* $\mathbf{A}$ *and* $\mathbf{B}$, *and the characteristic vectors of* $\mathbf{A}\cdot\times\mathbf{B}$ *are the direct products* $\mathbf{u}_i\cdot\times\mathbf{v}_j$[2]) *of the characteristic vectors of* $\mathbf{A}$ *and* $\mathbf{B}$.

According to a theorem of STÉPHANOS[3]) the characteristic roots of the nm-th order hypermatrix $\sum_p\sum_q c_{pq}\mathbf{A}^p\cdot\times\mathbf{B}^q$ are the nm numbers $\sum_p\sum_q a_i^p b_j^q$, where the a_i's are the characteristic roots of $\mathbf{A}$ and the b_j's are the characteristic roots of $\mathbf{B}$.

By means of Theorem B STÉPHANOS' theorem can be easily completed as follows:

Corollary II. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are symmetrical with the spectral decompositions*

$$\mathbf{A}=\mathbf{U}\langle a_1,\ldots,a_m\rangle\mathbf{U}^*=\sum_k a_k\mathbf{u}_k\mathbf{u}_k^*,\quad \mathbf{U}\mathbf{U}^*=\mathbf{E}_m, \tag{4.1}$$

$$\mathbf{B}=\mathbf{V}\langle b_1,\ldots,b_n\rangle\mathbf{V}^*=\sum_h b_h\mathbf{v}_h\mathbf{v}_h^*,\quad \mathbf{V}\mathbf{V}^*=\mathbf{E}_n, \tag{4.2}$$

then the spectral decomposition of $\sum_p\sum_q c_{pq}\mathbf{A}^p\cdot\times\mathbf{B}^q$ *is given by*

$$\sum_i\sum_j(\mathbf{u}_i\cdot\times\mathbf{v}_j)\Big(\sum_p\sum_q c_{pq}a_i^p b_j^q\Big)(\mathbf{u}_i^*\cdot\times\mathbf{v}_j^*). \tag{4}$$

3. Determinant of a hypermatrix. Let be given an nm-th order hypermatrix $[\mathbf{A}_{ii}]$ whose m-th order blocks $\mathbf{A}_{ij}$ are commutable in pairs. Suppose preliminarily that the hypermatrices

$$\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12} & \ldots & \mathbf{A}_{1k}\\ \mathbf{A}_{21} & \mathbf{A}_{22} & \ldots & \mathbf{A}_{2k}\\ \cdot & \cdot & \cdot & \cdot\\ \mathbf{A}_{k1} & \mathbf{A}_{k2} & \ldots & \mathbf{A}_{kk}\end{bmatrix}\qquad (k=1,2,\ldots,n-1) \tag{5}$$

are invertable.

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[2]) In agreement with the general definition we designate by $\mathbf{u}\cdot\times\mathbf{v}$ the vector

$$\mathbf{u}\cdot\times\mathbf{v}=\mathbf{u}\cdot\times\begin{bmatrix}v_1\\ v_2\\ \vdots\\ v_n\end{bmatrix}=\begin{bmatrix}\mathbf{u}v_1\\ \mathbf{u}v_2\\ \vdots\\ \mathbf{u}v_n\end{bmatrix}.$$

[3]) C. STÉPHANOS, Sur une extension du calcul des substitutions linéaires, *Journal math. pures et appliquées,* (5) **6** (1900), 73—126.

Then, replacing in the known scalar identity[4])

$$(6)\quad \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ b_{21} & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \dots & 1 \end{bmatrix} \begin{bmatrix} D_1 & 0 & \dots & 0 \\ 0 & D_2/D_1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & D_n/D_{n-1} \end{bmatrix} \begin{bmatrix} 1 & c_{12} & \dots & c_{1n} \\ 0 & 1 & \dots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1 \end{bmatrix},$$

where

$$D_k = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \cdot & \cdot & \cdot & \cdot \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{vmatrix} \neq 0; \quad b_{ik} = D_k^{-1} \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \cdot & \cdot & \cdot & \cdot \\ a_{k-1,1} & a_{k-1,2} & \dots & a_{k-1,k} \\ a_{i1} & a_{i2} & \dots & a_{ik} \end{vmatrix};$$

$$c_{kj} = D_k^{-1} \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1,k-1} & a_{1j} \\ a_{21} & a_{22} & \dots & a_{2,k-1} & a_{2j} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{k1} & a_{k2} & \dots & a_{k,k-1} & a_{kj} \end{vmatrix} \quad (k = 1, 2, \dots, n-1),$$

the scalar indeterminates a_{ij} by the commutable blocks $\mathbf{A}_{ij}$ and taking determinants on both sides, we get immediately

$$(7)\qquad \det \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \dots & \mathbf{A}_{1n} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \dots & \mathbf{A}_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \mathbf{A}_{n1} & \mathbf{A}_{n2} & \dots & \mathbf{A}_{nn} \end{bmatrix} = \det\Big(\sum_{(\nu)} \pm \mathbf{A}_{1\nu_1}\mathbf{A}_{2\nu_2}\dots\mathbf{A}_{n\nu_n}\Big).$$

Thus we have proved formula (1) of our Theorem A under the restrictive conditions (5), but it is easy to see for reasons of continuity that it holds in the general case too.

Introducing the term "hyperdeterminant" for

$$\mathbf{det}\,[\mathbf{A}_{ij}] = \sum_{(\nu)} \pm \mathbf{A}_{1\nu_1}\mathbf{A}_{2\nu_2}\dots\mathbf{A}_{n\nu_n}$$

we can formulate the following theorem.

If the blocks of a hypermatrix are commutable in pairs, then the determinant of the hypermatrix is equal to the (ordinary) determinant of the hyperdeterminant of the blocks.

Obviously this theorem reduces the computation of an mn-th order determinant to that of an n-th order determinant.

4. Adjoint of a hypermatrix. Attempting to find a similar method for the computation of the adjoint, at first sight it is not easy to see, how to utilize the commutability of the blocks, because the formation of a minor i. e. the deleting of a row and of a column destroys the partitioned structure of the hypermatrix. Nevertheless a very simple idea will enable us

[4]) See e. g. Ф. Я. Гантмахер, Теория матриц (Москва, 1953), p. 39.

to find a convenient method, namely, one must not endeavour to calculate a single element of the adjoint, but one has to find a whole block of it at once.

To this end, we introduce the concept of the "hyperadjoint" $\mathbf{adj}\,[\mathbf{A}_{ij}]$ of the given hypermatrix $[\mathbf{A}_{ij}]$ composed of commutable blocks. It is a hypermatrix of the same order as $[\mathbf{A}_{ij}]$, whose blocks depend in the same way upon the blocks $\mathbf{A}_{ij}$ as the elements of the ordinary adjoint depend upon the scalar elements of an ordinary matrix.

According to this definition the hyperadjoint satisfies the identity

$$(8)\qquad [\mathbf{A}_{ij}]\,\mathbf{adj}\,[\mathbf{A}_{ij}] = \begin{bmatrix} \mathbf{det}\,[\mathbf{A}_{ij}] & 0 & \dots & 0 \\ 0 & \mathbf{det}\,[\mathbf{A}_{ij}] & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \mathbf{det}\,[\mathbf{A}_{ij}] \end{bmatrix},$$

hence it is obvious that in order to have the adjoint of $[\mathbf{A}_{ij}]$ one has only to multiply from the right both sides of the equation (8) by a diagonal hypermatrix all of whose diagonal elements are equal to $\operatorname{adj}\mathbf{det}\,[\mathbf{A}_{ij}]$. We have indeed

$$\mathbf{det}\,[\mathbf{A}_{ij}]\cdot\operatorname{adj}\mathbf{det}\,[\mathbf{A}_{ij}] = \det\,[\mathbf{A}_{ij}]\cdot\mathbf{E}_m,$$

$$[\mathbf{A}_{ij}]\cdot\{\mathbf{adj}\,[\mathbf{A}_{ij}]\cdot\operatorname{adj}\mathbf{det}\,[\mathbf{A}_{ij}]\cdot\times\mathbf{E}_n)\} = \det\,[\mathbf{A}_{ij}]\cdot\mathbf{E}_{mn}.$$

Thus the adjoint of $[\mathbf{A}_{ij}]$ is equal to the matrix in brackets { }, and so formula (2) of our Theorem A is proved.

This result can be formulated as follows:

In order to find the adjoint of a hypermatrix whose blocks are commutable, one has to multiply each block of the hyperadjoint $\mathbf{adj}\,[\mathbf{A}_{ij}]$ *by* $\operatorname{adj}\mathbf{det}\,[\mathbf{A}_{ij}]$.

In this way the computation of the adjoint of an mn-th order hypermatrix, whose blocks are commutable, is reduced to operations with n-th order and m-th order matrices.

If the hypermatrix $[\mathbf{A}_{ij}]$ is non-singular then the expression (2) of the adjoint furnishes the following formula for the inverse of $[\mathbf{A}_{ij}]$

$$(9)\qquad [\mathbf{A}_{ij}]^{-1} = \mathbf{adj}\,[\mathbf{A}_{ij}]\,(\mathbf{det}\,[\mathbf{A}_{ij}]\cdot\times\mathbf{E}_n)^{-1}.$$

5. Special cases. The direct product of two quadratical matrices

$$\mathbf{A}\cdot\times\mathbf{B} = \begin{bmatrix} \mathbf{A}b_{11} & \mathbf{A}b_{12} & \dots & \mathbf{A}b_{1n} \\ \mathbf{A}b_{21} & \mathbf{A}b_{22} & \dots & \mathbf{A}b_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \mathbf{A}b_{n1} & \mathbf{A}b_{n2} & \dots & \mathbf{A}b_{nn} \end{bmatrix}$$

is the simplest instance of a hypermatrix whose blocks are commutable in pairs. Thus the properties of the direct product must be consequences of the theorems wich we proved above.

Indeed, application of (8) gives immediately

$$\det(\mathbf{A}\cdot\times\mathbf{B}) = \det\,[\mathbf{A}b_{ij}] = \det(\mathbf{det}[\mathbf{A}b_{ij}]) = \det(\mathbf{A}^n|\mathbf{B}|) = |\mathbf{A}|^n\cdot|\mathbf{B}|^m.$$

Further, denoting the cofactor of b_{ij} by B_{ij}, we have from (9)

$$(\mathbf{A}\cdot\times\mathbf{B})^{-1}=\mathbf{adj}\,[\mathbf{A}b_{ij}]\,(\mathbf{det}\,[\mathbf{A}b_{ij}]\cdot\times\mathbf{E}_n)^{-1}=$$

$$=[\mathbf{A}^{n-1}B_{ji}]\,|\mathbf{B}|^{-1}(\mathbf{A}^{-n}\cdot\times\mathbf{E}_n)=\left[\mathbf{A}^{-1}\frac{B_{ji}}{|\mathbf{B}|}\right]=\mathbf{A}^{-1}\cdot\times\mathbf{B}^{-1}.$$

W. VOIGT[5]) investigated the $2n$-th order hypermatrix $[\mathbf{A}_{ij}]$ where

$$\mathbf{A}_{ij}=\begin{bmatrix} b_{ij} & c_{ij}\\ -c_{ij} & b_{ij}\end{bmatrix}\qquad (i,j=1,2,\ldots,n),$$

and proved that the determinant of this hypermatrix is equal to the sum of two squares. VOIGT's theorem is an immediate consequence of our Theorem A, because the second-order blocks $\mathbf{A}_{ij}$ are skew-cyclic, hence commutable in pairs. Thus we have

$$\det\,[\mathbf{A}_{ij}]=\det\sum_{(\nu)}\pm\begin{bmatrix} b_{1\nu_1} & c_{1\nu_1}\\ -c_{1\nu_1} & b_{1\nu_1}\end{bmatrix}\begin{bmatrix} b_{2\nu_2} & c_{2\nu_2}\\ -c_{2\nu_2} & b_{2\nu_2}\end{bmatrix}\cdots\begin{bmatrix} b_{n\nu_n} & c_{n\nu_n}\\ -c_{n\nu_n} & b_{n\nu_n}\end{bmatrix}.$$

But sum and product of skew-cyclic matrices is again skew-cyclic, therefore

$$\det\,[\mathbf{A}_{ij}]=\det\begin{bmatrix} B & C\\ -C & B\end{bmatrix}=B^2+C^2$$

where B and C are rational integral functions of the elements b_{ij}, c_{ij}. Q. e. d.

6. Permutation matrix. We introduce a particular hypermatrix which we shall use several times in the subsequent sections. Let

$$\mathbf{e}_i=\begin{array}{c}\begin{bmatrix}0\\0\\ \vdots\\1\\ \vdots\\0\\0\end{bmatrix}\end{array}\!\!\begin{array}{l}(1\\ \\ \\(i\\ \\ \\(m\end{array}\quad\text{resp.}\quad \mathbf{f}_j=\begin{array}{c}\begin{bmatrix}0\\0\\ \vdots\\1\\ \vdots\\0\\0\end{bmatrix}\end{array}\!\!\begin{array}{l}(1\\ \\ \\(j\\ \\ \\(n\end{array}\qquad \begin{pmatrix} j=1,2,\ldots,m\\ j=1,2,\ldots,n\end{pmatrix}$$

be m-th resp. n-th order unit vectors and form the mn-th order quadratical hypermatrix

$$\text{(10)}\qquad \mathbf{P}=[\mathbf{e}_i\mathbf{f}_j^*]=\begin{bmatrix}\mathbf{e}_1\mathbf{f}_1^* & \mathbf{e}_2\mathbf{f}_1^* & \ldots & \mathbf{e}_m\mathbf{f}_1^*\\ \mathbf{e}_1\mathbf{f}_2^* & \mathbf{e}_2\mathbf{f}_2^* & \ldots & \mathbf{e}_m\mathbf{f}_2^*\\ \cdot & \cdot & \cdot & \cdot\\ \mathbf{e}_1\mathbf{f}_n^* & \mathbf{e}_2\mathbf{f}_n^* & \ldots & \mathbf{e}_m\mathbf{f}_n^*\end{bmatrix}.$$

The blocks of $\mathbf{P}$ and $\mathbf{P}^*$ are conformable and we have

$$\mathbf{P}\mathbf{P}^*=\mathbf{E}_{mn}$$

i. e. $\mathbf{P}$ is orthogonal. Furthermore it is easy to see that multiplication of the

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[5]) W. VOIGT, Allgemeine Formeln für die Bestimmung der Elasticitätskonstanten von Krystallen, *Wiedemanns Annalen Phys. Chem.*, 16 (1882), 273—321.

sequence (as a row-vector)

$$(11)(12)\dots(1n);\ (21)(22)\dots(2n);\ \dots;\ (m1)(m2)\dots(mn)$$

by $\mathbf{P}$ transforms it into

$$(11)(21)\dots(m1);\ (12)(22)\dots(m2);\ \dots;\ (1n)(2n)\dots(mn)$$

We will prove the relation

$$\mathbf{A}\cdot\times\mathbf{B}=\mathbf{P}(\mathbf{A}\times\cdot\mathbf{B})\mathbf{P}^* \tag{11}$$

which expresses that the left and right direct products of the same square matrices are orthogonally similar. To this end partition $\mathbf{A}$ in columns, $\mathbf{B}$ in rows:

$$\mathbf{A}=[\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_m],\quad \mathbf{B}=\begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \vdots\\ \mathbf{b}^n\end{bmatrix},$$

and consider the mn-th order square hypermatrix

$$[\mathbf{a}_i\mathbf{b}^j]=\begin{bmatrix}\mathbf{a}_1\mathbf{b}^1 & \mathbf{a}_2\mathbf{b}^1 \cdots & \mathbf{a}_m\mathbf{b}^1\\ \mathbf{a}_1\mathbf{b}^2 & \mathbf{a}_2\mathbf{b}^2 \cdots & \mathbf{a}_m\mathbf{b}^2\\ \vdots & \vdots & \vdots\\ \mathbf{a}_1\mathbf{b}^n & \mathbf{a}_2\mathbf{b}^n \cdots & \mathbf{a}_m\mathbf{b}^n\end{bmatrix}.$$

Multiplication of this matrix by $\mathbf{P}^*$ gives

$$[\mathbf{a}_i\mathbf{b}_j]\cdot\mathbf{P}^*=[\mathbf{a}_i\mathbf{b}^j][\mathbf{f}_k\mathbf{e}_i^*]=[\sum_\nu \mathbf{a}_\nu\mathbf{b}^j\mathbf{f}_k\mathbf{e}_\nu^*]=[b_{jk}\sum_\nu \mathbf{a}_\nu\mathbf{e}_\nu^*]=[b_{jk}\mathbf{A}]=\mathbf{A}\cdot\times\mathbf{B},$$

and similarly

$$\mathbf{P}^*[\mathbf{a}_i\mathbf{b}^j]=\mathbf{A}\times\cdot\mathbf{B}.$$

Hence

$$\mathbf{P}^*[\mathbf{a}_i\mathbf{b}^j]\mathbf{P}^*=(\mathbf{A}\times\cdot\mathbf{B})\mathbf{P}^*=\mathbf{P}^*(\mathbf{A}\cdot\times\mathbf{B})$$

or

$$\mathbf{A}\cdot\times\mathbf{B}=\mathbf{P}(\mathbf{A}\times\cdot\mathbf{B})\mathbf{P}^*.$$ Q. e. d.

7. Spectral decomposition of a symmetrical hypermatrix. We suppose from now on that the blocks $\mathbf{A}_{ij}$ have the form $\mathbf{A}_{ij}=f_{ij}(\mathbf{A})$, where $\mathbf{A}$ is an m-th order symmetrical matrix and $f_{ij}(x)$ are arbitrary polynomials of the real variable x subject only to the restriction $f_{ji}(x)\equiv\overline{f_{ij}(x)}$.

We write the spectral decomposition of a polynomial $f(\mathbf{A})$ of the symmetrical matrix $\mathbf{A}$ in the following form

$$f(\mathbf{A})=\mathbf{U}\langle f(a_k)\rangle\mathbf{U}^*=[\mathbf{u}_1,\mathbf{u}_2,\dots,\mathbf{u}_m]\langle f(a_1),f(a_2),\dots,f(a_m)\rangle\begin{bmatrix}\mathbf{u}_1^*\\ \mathbf{u}_2^*\\ \vdots\\ \mathbf{u}_m^*\end{bmatrix}= \tag{12}$$

$$=\sum_k f(a_k)\mathbf{u}_k\mathbf{u}_k^*$$

or

$$\mathbf{U}^*f(\mathbf{A})\mathbf{U}=\langle f(a_1),f(a_2),\dots,f(a_m);\quad \mathbf{U}\mathbf{U}^*=\mathbf{E}_m. \tag{12.1}$$

Here $a_1, a_2, \ldots, a_m$ are the (real) characteristic roots and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ the characteristic vectors of $\mathbf{A}$.

Transform now the given hypermatrix $[\mathbf{A}_{ij}] = [f_{ij}(\mathbf{A})]$ as follows:

$$(\mathbf{U}^* \cdot \times \mathbf{E}_n)[f_{ij}(\mathbf{A})](\mathbf{U} \cdot \times \mathbf{E}_n) =$$

$$= \begin{bmatrix} \mathbf{U}^* & 0 & \ldots & 0 \\ 0 & \mathbf{U}^* & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & \mathbf{U}^* \end{bmatrix} \begin{bmatrix} f_{11}(\mathbf{A}) & f_{12}(\mathbf{A}) & \ldots & f_{1n}(\mathbf{A}) \\ \cdot & \cdot & \cdot & \cdot \\ f_{n1}(\mathbf{A}) & f_{n2}(\mathbf{A}) & \ldots & f_{nn}(\mathbf{A}) \end{bmatrix} \begin{bmatrix} \mathbf{U} & 0 & \ldots & 0 \\ 0 & \mathbf{U} & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & \mathbf{U} \end{bmatrix} =$$

$$= \begin{bmatrix} \langle f_{11}(a_k)\rangle & \langle f_{12}(a_k)\rangle & \ldots & \langle f_{1n}(a_k)\rangle \\ \langle f_{21}(a_k)\rangle & \langle f_{22}(a_k)\rangle & \ldots & \langle f_{2n}(a_k)\rangle \\ \cdot & \cdot & \cdot & \cdot \\ \langle f_{n1}(a_k)\rangle & \langle f_{n2}(a_k)\rangle & \ldots & \langle f_{nn}(a_k)\rangle \end{bmatrix}.$$

Using the permutation matrix $\mathbf{P}$ we get from here

$$\mathbf{P}^*(\mathbf{U}^* \cdot \times \mathbf{E}_n)[f_{ij}(\mathbf{A})](\mathbf{U} \cdot \times \mathbf{E}_n)\mathbf{P} =$$

$$= \left\langle \begin{bmatrix} f_{11}(a_1) & \ldots & f_{n1}(a_1) \\ \cdot & \cdot & \cdot \\ f_{1n}(a_1) & \ldots & f_{nn}(a_1) \end{bmatrix}, \ldots, \begin{bmatrix} f_{11}(a_m) & \ldots & f_{n1}(a_m) \\ \cdot & \cdot & \cdot \\ f_{n1}(a_m) & \ldots & f_{nn}(a_m) \end{bmatrix} \right\rangle.$$

In consequence of our assumption $f_{ji} \equiv \overline{f_{ij}}$ each diagonal block on the right is symmetrical, therefore each block $[f_{ij}(a_k)]$ can be diagonalized by the aid of an orthogonal matrix $\mathbf{V}_k$ in the form

$$\mathbf{V}_k^* \begin{bmatrix} f_{11}(a_k) & \ldots & f_{1n}(a_k) \\ \cdot & \cdot & \cdot \\ f_{n1}(a_k) & \ldots & f_{nn}(a_k) \end{bmatrix} \mathbf{V}_k = \langle \lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kn} \rangle. \tag{12.2}$$

Thus the spectral decomposition of $[f_{ij}(\mathbf{A})]$ is given by the equation

$$\mathbf{T}^*[f_{ij}(\mathbf{A})]\mathbf{T} = \langle \lambda_{11}, \ldots, \lambda_{1n}, \lambda_{21}, \ldots, \lambda_{2n}, \ldots, \lambda_{m1}, \ldots, \lambda_{mn} \rangle \tag{12.21}$$

where

$$\mathbf{T}^* = (\mathbf{U} \cdot \times \mathbf{E}_n)\mathbf{P}\langle \mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_m \rangle, \quad \mathbf{T}\mathbf{T}^* = \mathbf{E}_{mn}.$$

8. Spectral decomposition of a direct product. Let us consider the simplest case where the f_{ij} are homogeneous linear functions, i. e. $f_{ij}(x) = b_{ij}x$ $(b_{ji} = \overline{b_{ij}})$. Then the hypermatrix $[f_{ij}(\mathbf{A})]$ is a direct product:

$$[f_{ij}(\mathbf{A})] = [\mathbf{A} b_{ij}] = \mathbf{A} \cdot \times \mathbf{B} \qquad (\mathbf{B} = [b_{ij}]).$$

In this case the spectral decomposition (12.21) will become particularly simple. Indeed, let the spectral decomposition of $\mathbf{B}$ be

$$\mathbf{V}^*\mathbf{B}\mathbf{V} = \langle b_1, b_2, \ldots, b_n \rangle, \quad \mathbf{V}\mathbf{V}^* = \mathbf{E}_n, \tag{12.3}$$

then each block $[f_{ij}(a_k)] = a_k[b_{ij}]$ can be diagonalized by the same transforming matrix $\mathbf{V}$ in the form

$$\mathbf{V}^{*}\begin{bmatrix} a_k b_{11} & \dots & a_k b_{1n} \\ \cdot & \cdot & \cdot \\ a_k b_{n1} & \dots & a_k b_{nn} \end{bmatrix}\mathbf{V} = a_k \langle b_1, b_2, \dots, b_n \rangle.$$

The equation (12.21) reduces after the substitution $\mathbf{V}_1 = \dots = \mathbf{V}_k = \mathbf{V}$ to

$$(\mathbf{V}^{*}\cdot\times\mathbf{E}_n)\mathbf{P}^{*}(\mathbf{U}^{*}\cdot\times\mathbf{E}_m)(\mathbf{A}\cdot\times\mathbf{B})(\mathbf{U}\cdot\times\mathbf{E}_m)\mathbf{P}(\mathbf{V}\cdot\times\mathbf{E}_n) =$$
$$= \langle a_1 b_1, \dots, a_1 b_n, a_2 b_1, \dots, a_2 b_n, \dots, a_m b_1, \dots, a_m b_n \rangle.$$

Finally, using once more the permutation matrix $\mathbf{P}$, we get from here

$$\mathbf{A}\cdot\times\mathbf{B} = (\mathbf{U}\cdot\times\mathbf{V})\langle a_1 b_1, \dots, a_m b_1, \dots, a_1 b_n, \dots, a_m b_n \rangle(\mathbf{U}^{*}\cdot\times\mathbf{V}^{*}) =$$
$$= \sum_i \sum_j (\mathbf{u}_i\cdot\times\mathbf{v}_j)(a_i b_j)(\mathbf{u}_i^{*}\cdot\times\mathbf{v}_j^{*}).$$

Thus we have proved that if the hypermatrix is a direct product of two symmetrical matrices then the characteristic roots are the products of the characteristic roots of the factors and the characteristic vectors are the direct products of the characteristic vectors of the factors.

From here we deduce by straightforward application of the operational rules for direct products the following completition of a theorem of STÉPHANOS:

If the spectral decomposition of the symmetrical matrices $\mathbf{A}$ and $\mathbf{B}$ is given by the equations (12.1, 12.3) then

$$\sum_p \sum_q c_{pq}\mathbf{A}^p\cdot\times\mathbf{B}^q = \sum_i \sum_j (\mathbf{u}_i\cdot\times\mathbf{v}_j)\Big(\sum_p \sum_q c_{pq} a_i^p b_j^q\Big)(\mathbf{u}_i^{*}\cdot\times\mathbf{v}_j^{*}). \tag{13}$$

We note for later application the particular case of (13) corresponding to

$$c_{10} = c_{01} = 1 \text{ and all other } c_{pq} = 0:$$

$$\mathbf{A}\cdot\times\mathbf{E} + \mathbf{E}\cdot\times\mathbf{B} = \sum_i \sum_j (\mathbf{u}_i\cdot\times\mathbf{v}_j)(a_i + b_j)(\mathbf{u}_i^{*}\cdot\times\mathbf{v}_j^{*}). \tag{13.1}$$

9. Application. There is no doubt that the concepts direct product as well as that of the hypermatrix owe their origin to grouptheoretical or other abstract considerations. Nevertheless these concepts turn out to be the appropriate tools for the mathematical investigation of various mechanical systems which possess regular structure. We confine ourselves here to show their application in lattice-dynamics.

In lattice-dynamics systems of particles are investigated which, in their equilibrium position, form a regular lattice, each particle being acted upon by its nearest neighbours and perhaps by a fixed boundary.

The knowledge of the normal vibrations of a finite lattice of particles is of prime importance in the corpuscular theory of matter, thus the equations of motion of a finite lattice have been investigated by several writers. In spite of these efforts, at present — as far as the writer is informed — it is only the one-dimensional lattice of equal equidistant particles whose normal vibrations are known. In the case of two- or three-dimensional lattices only the periods of the normal vibrations, i. e. the characteristic roots have been calculated[6]).

The application of Theorem B will enable us to determine the normal vibrations of a two- or three-dimensional lattice of particles with fixed boundary. The result is surprisingly simple: The characteristic vectors of a two- or three-dimensional lattice turn out to be the direct products of the characteristic vectors of the one-dimensional "edge-lattices", while the squares of the frequencies are equal to the sum or the squares of the frequencies of the one-dimensional edge-lattices.

Thus the analogy between the relation of a string to a membran and the relation of their lattice-models in complete.

In order to construct the spectral decomposition of the matrix belonging to a two-dimensional rectangular lattice with fixed boundary, we start with the well-known spectral decomposition of the matrix belonging to a one-dimensional lattice composed of m equal and equidistant particles and fixed boundary points[7])

$$\mathbf{C}_m=\begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}=\mathbf{S}_m\left\langle 4\sin^2\frac{k\pi}{2m+2}\right\rangle\mathbf{S}_m=$$

$$(14)\qquad =\sum_k 4\sin^2\frac{k\pi}{2m+2}\,\mathbf{u}_k\mathbf{u}_k^*;$$

$$\mathbf{S}_m=[\mathbf{u}_1,\mathbf{u}_2,\dots,\mathbf{u}_m];\quad \mathbf{u}_i=\sqrt{\frac{2}{m+1}}\begin{bmatrix}\sin\frac{i\pi}{m+1}\\ \vdots \\ \sin\frac{2i\pi}{m+1}\\ \vdots \\ \sin\frac{mi\pi}{m+1}\end{bmatrix}\qquad (i=1,2,\dots,m);$$

$$\mathbf{S}_m^*=\mathbf{S}_m;\ \mathbf{S}_m^2=\mathbf{E}_m.$$

The mn-th order matrix which belongs to a two-dimensional rectangular lattice composed of mn equal particles and fixed boundaries can be written in the following forms[8])

[6]) See e. g. D. E. Rutherford, Some Continuant Determinants arising in Physics and Chemistry, *Proc. Royal Soc. Edinburgh*, **62** (1947), 229—236; **63** (1952), 232—241.

[7]) See l. c. [6]).

[8]) See l. c. [6]).

$$\begin{bmatrix} 4 & -1 & & & -1 & & & & & \\ -1 & 4 & & & & -1 & & & & \\ & & \ddots & & & & \ddots & & & \\ & & & 4 & & & & -1 & & \\ -1 & & & & 4 & -1 & & & & \\ & -1 & & & -1 & 4 & & & & \\ & & \ddots & & & & \ddots & & & \\ & & & -1 & & & & 4 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & 4 & -1 \\ & & & & & & & & -1 & 4 \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{C}_m & & & \\ & \mathbf{C}_m & & \\ & & \ddots & \\ & & & \mathbf{C}_m \end{bmatrix} + \begin{bmatrix} 2\mathbf{E}_m & -\mathbf{E}_m & & \\ -\mathbf{E}_m & 2\mathbf{E}_m & \ddots & \\ & \ddots & \ddots & -\mathbf{E}_m \\ & & -\mathbf{E}_m & 2\mathbf{E}_m \end{bmatrix} = \mathbf{C}_m \cdot \times \mathbf{E}_n + \mathbf{E}_m \cdot \times \mathbf{C}_n .$$

Application of the formulae (13, 13. 1) and the spectral decomposition (14) of $\mathbf{C}_n$ gives immediately

$$\mathbf{C}_m \cdot \times \mathbf{E}_n + \mathbf{E}_m \cdot \times \mathbf{C}_n = (\mathbf{S}_m \cdot \times \mathbf{S}_n) \left\langle 4\sin^2 \frac{k\pi}{2m+2} + 4\sin^2 \frac{l\pi}{2n+2} \right\rangle (\mathbf{S}_m \cdot \times \mathbf{S}_n),$$

$$(\mathbf{S}_m \cdot \times \mathbf{S}_n)^* = \mathbf{S}_m \cdot \times \mathbf{S}_n ; \quad (\mathbf{S}_m \cdot \times \mathbf{S}_n)^2 = \mathbf{E}_{mn} .$$

If we write the matrices $\mathbf{S}_m$ and $\mathbf{S}_n$ in the partitioned form

$$\mathbf{S}_m = [\mathbf{u}_1, \ldots, \mathbf{u}_m]; \quad \mathbf{S}_n = [\mathbf{v}_1, \ldots, \mathbf{v}_n]; \quad \mathbf{v}_j^* = \sqrt{\frac{2}{n+1}} \left[\sin \frac{j\pi}{n+1}, \ldots, \sin \frac{nj\pi}{n+1} \right],$$

then the columns of the transforming matrix $\mathbf{S}_m \cdot \times \mathbf{S}_n$ i. e. the characteristic vectors of the lattice will be given by

$$\mathbf{S}_m \cdot \times \mathbf{S}_n = [\mathbf{u}_1 \cdot \times \mathbf{v}_1, \ldots, \mathbf{u}_i \cdot \times \mathbf{v}_j, \ldots, \mathbf{u}_m \cdot \times \mathbf{v}_n] \qquad \begin{pmatrix} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n \end{pmatrix},$$

where

$$\mathbf{u}_i \cdot \times \mathbf{v}_j = \sqrt{\frac{2}{m+1}} \begin{bmatrix} \sin \frac{i\pi}{m+1} \\ \vdots \\ \sin \frac{mi\pi}{m+1} \end{bmatrix} \cdot \times \sqrt{\frac{2}{n+1}} \begin{bmatrix} \sin \frac{j\pi}{n+1} \\ \vdots \\ \sin \frac{nj\pi}{n+1} \end{bmatrix} =$$

$$= \frac{2}{\sqrt{(m+1)(n+1)}} \begin{bmatrix} \sin \frac{i\pi}{m+1} \sin \frac{j\pi}{n+1} \\ \sin \frac{2i\pi}{m+1} \sin \frac{j\pi}{n+1} \\ \vdots \\ \sin \frac{mi\pi}{m+1} \sin \frac{nj\pi}{n+1} \end{bmatrix} .$$

Thus we proved that the squares of the frequencies of the two-dimensional lattice are equal to the sums of squares of the frequencies of the one-dimensional edge-lattices and the characteristic vectors are equal to the direct products $\mathbf{u}_i \cdot \times \mathbf{v}_j$ of the characteristic vectors of the edge-lattices.

Applying the same procedure to the hypermatrix

$$\mathbf{C}_m \cdot \times \mathbf{E}_n \cdot \times \mathbf{E}_r + \mathbf{E}_m \cdot \times \mathbf{C}_n \cdot \times \mathbf{E}_r + \mathbf{E}_m \cdot \times \mathbf{E}_n \cdot \times \mathbf{C}_r .$$

one obviously arrives to the spectral decomposition of the matrix belonging to a three-dimensional rectangular lattice with fixed boundary.

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